CORRECTIONS TO THE MOTION OF A SYSTEM WITH TWO DEGREES OF FREEDOM HAVING ONE CYCLIC COORDINATE

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The problem of designing a control $u(t)$ which brings a system with two degrees of freedom and a cyclic coordinate to a prescribed stable steady-state motion is considered. The problem is solved for the case of small initial deviations of the system from the prescribed motion.

1. Let us consider a controlled system with two degrees of freedom described by the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i} + b_i u \qquad (i = 1, 2) \tag{1.1}$$

Here the q_i are the generalized coordinates, $T(q,q')$ is the kinetic energy, $\Pi(q)$ is the potential energy, and $b_i(q,q')$ are functions determining the direction of the external control. The functions T, Π, and b_i are assumed to be analytic. We shall take the coordinate q_2 in system (1.1) to be cyclic ([1], p. 344).

Selecting the quantities q_1, q_1', and $p_2 = \partial T/\partial q_2'$ as the fundamental variables, we write Equation (1.1) in the form

$$f\,[q_1'', q_1', q_1, p_2] = b_1\,(q_1, q_1', p_2)\,u, \quad dp_2\,/\,dt = b_2\,(q_1, q_1', p_2)\,u \tag{1.2}$$

For $u(t) \equiv 0$ let the steady-state motion of system (1.2) be stable in the linear approximation

$$q_1 = q_1^\circ = \text{const}, \qquad q_1' = 0, \qquad p_2 = p_2^\circ = \text{const} \tag{1.3}$$

The problem is to choose the control $u(t)$ which will bring system (1.2) to the prescribed motion (1.3). The initial deviations $\Delta q_1(0)$, $\Delta q_1'(0)$, and $\Delta p_2(0)$ from the prescribed motion (1.3) are assumed to be small.

2. Let us first consider the problem in the linear approximation. We shall assume that the characteristic equation of the first approximation of system (1.2) around the point (1.3) (when $u \equiv 0$) has, apart from a zero root corresponding to the integral of $p_2 = \text{const}$, two pure imaginary roots $\pm i\kappa$ (the case when this equation has three zero roots is unusual and will not be considered here).

Putting $\Delta q_1 = x_1$, $\Delta q_1' / \varkappa = x_2$, and $\Delta p_2 = x_3$ and changing the time scale to $\tau = \kappa t$, we reduce the equations of the first approximation of system (1.2) to the form

$$\frac{dx_1}{d\tau} = x_2, \qquad \frac{dx_2}{d\tau} = -x_1 + \gamma x_3 + \beta_1 u, \qquad \frac{dx_3}{d\tau} = \beta_2 u \tag{2.1}$$

The following control problem is studied for system (2.1).

Problem 2.1. Find the control $u^0(t)$ which takes system (2.1) from the state $x_i(0) = x_{i0}$ to the state $x_i(\tau^0) = 0$ $(i = 1, 2, 3)$ subject to the condition

$$\max_\vartheta (|u(\vartheta)| \text{ when } 0 \leqslant \vartheta \leqslant \tau^0) = \min_u \tag{2.2}$$

In order to simplify the calculations we choose the control time τ^0 to be multiple of the period 2π of the natural oscillations of system (2.1), i.e., $\tau^0 = k \cdot 2\pi$, where k is an integer.

Problem 2.1 is an optimal control problem. It can be solved by any one of the well-known methods in the theory of optimal processes. The aim of the present paper is to examine the solution which is based on the arguments proposed for a similar problem in [2].

We shall take is that system (2.1) is completely controllable [3] since in this case we can solve both the linear Problem 2.1 of control [2,3] under any x_{i0} and the original non-linear problem [4] for all small initial deviations Δq_1, $\Delta q_1'$, Δp_2. For system (2.1) to be completely controllable it is necessary and sufficient [3] that the vectors

$$\begin{pmatrix} 0 \\ \beta_1 \\ \beta_2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \gamma\beta_2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \gamma\beta_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\beta_2 \\ -\beta_1 \\ 0 \end{pmatrix}$$

be linearly independent. This condition is satisfied if and only if $\beta_2 \neq 0$ and $\{\gamma \neq 0$ or $\beta_1 \neq 0\}$, which we shall assume.

3. To solve Problem 2.1 by the procedure described in [2] we should set up the fundamental matrix $F(t)$ of the homogeneous system (2.1) and find

$$\alpha = \max\left(\sum_{i=1}^{3} l_i c_i\right) \tag{3.1}$$

for

$$\int_0^{\tau^\circ} \left| \sum_{i=1}^{3} l_i h_i(\vartheta) \right| d\vartheta = 1 \qquad \begin{pmatrix} h(\vartheta) = F(\tau^\circ - \vartheta)s, \ s = \{0, \beta_1, \beta_2\} \\ c = -F(\tau^\circ)x^\circ, \ x^\circ = \{x_{10}, x_{20}, x_{30}\} \end{pmatrix} \tag{3.2}$$

Let α^0 and l_i^0 be the solutions of problem (3.1)-(3.2). Then, the optimal control $u^\circ(t)$, which solves Problem 2.1 is determined by the equality

$$u^\circ(t) = \alpha^\circ \operatorname{sign} \left(\sum_{i=1}^{3} l_i^\circ h_i(t) \right) \tag{3.3}$$

In the given case the matrix $F(t)$ has the form

$$F(t) = \begin{pmatrix} \cos t & \sin t & \gamma(1 - \cos t) \\ -\sin t & \cos t & \gamma \sin t \\ 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

Therefore, by the nonsingular linear substitution

$$\lambda_1 = -\beta_1 l_1 - \beta_2 \gamma l_2, \quad \lambda_2 = -\beta_2 \gamma l_1 + \beta_1 l_2, \quad \lambda_3 = \beta_2 \gamma l_1 + \beta_2 l_3$$

problem (3.1)-(3.2) is transformed to the problem

$$\alpha = \max \left(\sum_{i=1}^{3} c_i^* \lambda_i \right) \tag{3.5}$$

with

$$\int_0^{\tau^\circ} |\lambda_1 \sin \vartheta + \lambda_2 \cos \vartheta + \lambda_3| \, d\vartheta = 1 \tag{3.6}$$

where

$$\begin{aligned} c_1^* &= -\frac{\beta_1}{\beta_1^2 + \beta_2^2\gamma^2} c_1 - \frac{\beta_2\gamma}{\beta_1^2 + \beta_2^2\gamma^2} c_2 + \frac{\beta_1\gamma}{\beta_1^2 + \beta_2^2\gamma^2} c_3 \\ c_2^* &= -\frac{\beta_2\gamma}{\beta_1^2 + \beta_2^2\gamma^2} c_1 + \frac{\beta_1}{\beta_1^2 + \beta_2^2\gamma^2} c_2 + \frac{\beta_2\gamma^2}{\beta_1^2 + \beta_2^2\gamma^2} c_3 \\ c_3^* &= \frac{1}{\beta_2} c_3 \end{aligned}$$

The transformation $\{l_i\} \leftrightarrow \{\lambda_i\}$ is nonsingular as a consequence of the complete controllability of system (2.1). Indeed, otherwise it would be possible to find a nonzero l_i for which the integrand in (3.2) would be identically zero. This is impossible in the case of complete controllability [2, 3].

Thus, to determine the optimal control $u^0(t)$ in (3.3) we should solve problem (3.5)-(3.6).

Problem (3.5)-(3.6) is solved by well-known methods of differential calculus. To do this it is necessary to write the equation of the surface (3.6) in the $\{\lambda_i\}$ ($i = 1, 2, 3$) in explicit form. Equation (3.6) corresponds to a surface of rotation around the λ_3-axis. It is symmetric with respect to the plane $\lambda_3 = 0$. Therefore, it is sufficient to find the cross-section of this surface by the plane $\lambda_2 = 0$ in the first quadrant (Fig. 1).

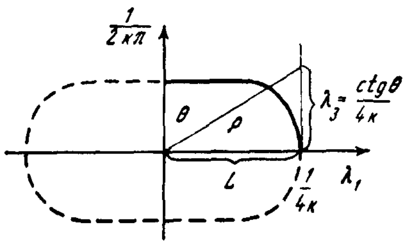

FIG. 1.

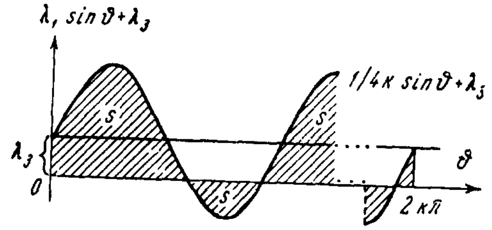

FIG. 2.

By geometric reasoning it follows that on the curve under consideration (Figs. 1,2)

$$\rho = \frac{L}{S(\lambda_3)\sin\theta}, \quad S(\lambda_3) = \int_0^{2k\pi} \left| \frac{1}{4k}\sin\vartheta + \lambda_3 \right| d\vartheta \quad \text{when } \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$$

$$\lambda_3 = \frac{1}{2k\pi} \quad \text{when } 0 \leqslant \lambda_1 \leqslant \frac{1}{2k\pi}$$

Therefore, in the spherical coordinates ρ, θ, ϕ, the surface (3.6) is described by the equations

$$\rho = \frac{1}{2k\pi\cos\theta} \quad \text{when } 0 \leqslant \theta \leqslant \frac{\pi}{4} \tag{3.7}$$

$$\rho = \frac{1}{4k\,(\sqrt{1-\cot^2\theta} + \arcsin\cot\theta\cdot\cot\theta)\sin\theta} \quad \text{when } \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2} \tag{3.8}$$

After constructing the curves (3.7) and (3.8), problem (3.5)-(3.6) is easily solved graphically. The equation

$$\sum_{i=1}^{3} \lambda_i c_i^* = \beta \qquad (-\infty < \beta < +\infty) \tag{3.9}$$

describes a family of parallel planes in the $\{\lambda_i\}$. Therefore, the numbers λ_i^0 which solve problem (3.5)-(3.6) are determined as the coordinates of the point $\{\lambda_i^0\}$ where the plane in (3.9) for $\beta = \alpha^0 > 0$ is tangent to the surface (3.6). This point is conveniently found by considering the curves (3.7) and (3.8) in the plane (Fig. 3) which is perpendicular to the lines of intersection of the planes (3.9) and $\lambda_3 = 0$.

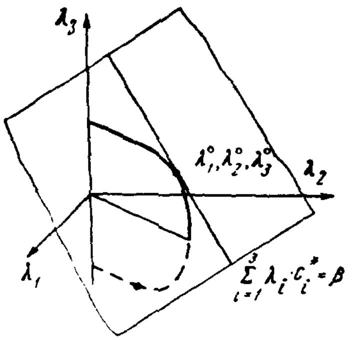

FIG. 3.

The only exceptional case is when $c_1^* = c_2^* = 0$. In this case we have $(\lambda_1^\circ)^2 + (\lambda_2^\circ)^2 \leqslant (\lambda_3^\circ)^2$, and the optimal control $u^\circ(t)$ in (3.3) maintains a constant sign for all θ in $[0, r^\circ]$. We shall not consider this case here. Let us assume that the initial deviations x_{i0} satisfy the condition $(c_1^*)^2 + (c_2^*)^2 > 0$. Then, the geometric reasoning described above determines in a unique manner the values of λ_i° which lie in the region

$$\lambda_1^2 + \lambda_2^2 > \lambda_3^2 \tag{3.10}$$

Here the optimal (3.3) is bang-bang. In the neighbourhood of any point $\{\lambda_i\}$ from region (3.10) both the principal curvatures of surface (3.6) are positive. This is verified, for example, by starting with Equations (3.7) and (3.8). Hence it follows that small changes Δx_{i0} in the quantities x_{i0}, or small changes Δc_i^* in the coefficients c_i^* of the planes (3.9), give rise to small changes $\Delta\lambda_i^\circ$ in the quantities λ_i°. Here, for every pair of numbers $\delta > 0$ and $\epsilon > 0$ we can find a number N such that

$$|\Delta\lambda_i^\circ| \leqslant N \|\Delta c^*\| \qquad (i = 1, 2, 3) \tag{3.11}$$

if we consider only those values of c_i^* lying in the region

$$(c_1^*)^2 + (c_2^*)^2 \geqslant \delta, \qquad \|c^*\| \leqslant \varepsilon \tag{3.12}$$

(the symbol $\|q\|$ denotes the euclidean norm of the vector q). Relying on this fact we arrive at the following conclusion.

Theorem 3.1. The optimal control $u^\circ(t)$ which solves Problem 2.1 has the form

$$u^\circ(t) = \alpha^\circ(x^\circ) \operatorname{sign}(\lambda_1^\circ \sin t + \lambda_2^\circ \cos t + \lambda_3^\circ) \tag{3.13}$$

Here $\alpha^\circ(x^\circ)$ and $\{\lambda_i^\circ\}$ are the solutions of problem (3.5)-(3.6) and, moreover, the surface (3.6) is determined by equations (3.7) and (3.8). The estimate

$$\alpha^\circ(x^\circ) \leqslant N_1 \|c^*\| \tag{3.14}$$

is valid.

Small change Δx_{i0} in the quantities x_{i0} give rise to changes $\Delta u^\circ(t)$ in the optimal control which are small on the average. Under conditions (3.11) and (3.12) the estimates

$$|\Delta\alpha^\circ| \leqslant N_2 \|\Delta c^*\|, \qquad |\Delta u^\circ(t)| \leqslant N_3 \|\Delta c^*\| \tag{3.15}$$

are valid for all t except for a set Q of values of t whose measure $\mu(Q)$ satisfies the inequality

$$\mu(Q) \leqslant N_4 \|\Delta c^*\| \qquad (N_i\ (i = 1, \ldots, 4), \text{ are constants}) \tag{3.16}$$

Remark 3.1. The exclusion of the case $c_1^* = 0$, $c_2^* = 0$ does not raise any serious problems. However, a consideration of this case requires a subsidiary investigation of the nature of the smoothness of the surfaces (3.7) and (3.8) at the point $\theta = \pi/4$, which is outside the scope of the present paper.

4. Let us now consider the question of control in the nonlinear system (1.2). Taking as a start the control $u^0(t)$ in (3.3) found in the linear approximation for Problem 2.1, we can construct an iterative process for the determination of a certain control $u^0(t)$ which solves the nonlinear problem. Here, the nonlinear terms of all the higher orders are taken into account at each step.

Let us describe the first step of the iterative process after the solution of Problem 2.1. Let us denote the control $u^0(t)$ from (3.3), which solves Problem 2.1 subject to some initial condition $x^\circ = \{x_{10}, x_{20}, x_{30}\}$, by the symbol $u_{(1)}^\circ(t, x^\circ)$. If in equation (1.2) we take into account terms of the second order of smallness in Δq_1, $\Delta q_1'$, Δp_2 and u, then, in the variables $x_i(t)$, we obtain the system

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 + \gamma x_3 + \beta_1 u + f_1^{(2)}(x, u), \quad \frac{dx_3}{dt} = \beta_2 u + f_2^{(2)}(x, u) \qquad (4.1)$$

Here the functions $f_1^{(2)}(x, u)$ and $f_2^{(2)}(x, u)$ are second-order forms in their arguments. In equation (4.1) let us substitute the value of the control $u = u_{(1)}^\circ(t, x^\circ)$. This control (3.13) satisfies estimate (3.14) and, consequently, also the estimate

$$| u_{(1)}^\circ(t, x^\circ)| \leqslant N_5 \| x^\circ \| \quad (N_5 = \text{const}) \qquad (4.2)$$

In the linear approximation (2.1) the control $u_{(1)}^\circ(t, x^\circ)$ transfers system (4.1) to the equilibrium state $x(\tau^0) = 0$ and, in addition, as a consequence of estimate (4.2) the motion $x(t, x^\circ)_{(2.1)}$, of system (2.1) satisfies the inequality

$$\| x(t, x^\circ)_{(2.1)} \| \leqslant N_6 \| x^\circ \| \quad (N_6 = \text{const}) \qquad (4.3)$$

Hence, from the well-known properties of ordinary differential equations [5] we conclude that when $u = u_{(1)}^\circ(t, x^\circ)$ the motion $x(t, x^\circ)_{(4.1)}$ of system (4.1) is led to the state

$$x(\tau^\circ, x^\circ)_{(4.1)} = y^{(1)}, \qquad \| y^{(1)} \| \leqslant N_7 \| x^\circ \|^2 \qquad (4.4)$$

With an accuracy upto terms of the third order of smallness in x^0 the vector $y^{(1)}$ is

$$y^{(1)} = \int_0^{\tau^\circ} F(\tau^\circ - \vartheta) f^{(2)}(\vartheta)\, d\vartheta \qquad (4.5)$$

$$f^{(2)}(\vartheta) = \{0, f_1^{(2)}(x(\vartheta, x^\circ)_{(2.1)}, u_{(1)}^\circ(\vartheta, x^\circ)), f_2^{(2)}(x(\vartheta, x^\circ)_{(2.1)}, u_{(1)}^\circ(\vartheta, x^\circ))\}$$

Here the function $F(t)$ is the fundamental matrix of system (2.1). In order to consider and correct the magnitude of the error $x\ (\tau^\circ,\ x^\circ)_{(4.1)} = y^{(1)}$ (4.4) and (4.5), let us consider anew the control of Problem 2.1 in the linear approximation but now not to the point $x(\tau^0) = 0$ but to the point $x\ (\tau^\circ) = -\ y^{(1)}$. This problem again reduces to problem (3.1)-(3.2) where the vector c is now defined by the equality $c = -\ F\ (\tau^\circ)\ x^\circ - y^{(1)}$. A change Δc in the vector c causes a corresponding change Δc^* of the very same order in the vector c^* of problem (3.5)-(3.6). Let $u_{(2)}{}^\circ\ (t,\ x^\circ)$ be the control which is obtained corresponding to equality (3.3) for problem (3.5)-(3.6) altered in the manner described. From Theorem 3.1 it follows that the change $\Delta u^\circ = u_{(2)}{}^\circ\ (t,\ x^\circ) - u_{(1)}{}^\circ\ (t,\ x^\circ)$ in the control $u^0(t)$ will on the average be of the second order of smallness in x^0.

Namely, the estimate

$$|\ \Delta u^\circ\ (t)\ | \leqslant N_8\ \|\ y^{(1)}\ \| = N_7 N_8\ \|\ x^\circ\ \|^2 \tag{4.6}$$

will be valid for all values of t with the exception of a set $Q^{(1)}(x^0)$ of values of t whose measure $\mu(Q^{(1)})$ satisfies the inequality

$$\mu\ (Q^{(1)}\ (x^\circ)) \leqslant N_9\|\ y^{(1)}\ \| = N_7 N_9\|\ x^\circ\ \|^2 \tag{4.7}$$

and, in addition,

$$|\ u_{(2)}{}^\circ\ (t,\ x^\circ)\ | \leqslant N_{10}\ \|x^\circ\ \| \qquad (N_7,\ \ldots,\ N_{10} = \text{const}) \tag{4.8}$$

The control $u_{(2)}{}^\circ\ (t,\ x^\circ)$ can be chosen as the second approximation for the solution of the original nonlinear control problem. By such a choice of control and from estimates (4.6)-(4.8) it follows that the control $u_{(2)}{}^\circ\ (t,\ x^\circ)$ transfers system (4.1), and also the system (1.2), to the state $x\ (\tau^\circ,\ x^\circ) = y^{(2)}$, where the vector $y^{(2)}$ is of the third order of smallness in $\|x^0\|$. With the help of vector $y^{(2)}$ we can construct a new approximation $u_{(3)}{}^\circ\ (t,\ x^\circ)$, in the same way as the approximation $u_{(2)}{}^\circ\ (t,\ x^\circ)$ was constructed from the vector $y^{(1)}$ in (4.5), etc. The estimates mentioned in Theorem 3.1 ensure, at every step, an increase in the order of the vector $y^{(k)}$ and, by the same token, give an estimate of the convergence of the iteration process.

5. Let us consider an example. Suppose we are given a mathematical pendulum (Fig. 4) whose horizontal axis of suspension $\{O_1, O_1\}$ rotates around the vertical axis $\{O_{x4}\}$.

This rotation is controlled by a moment $u(t)$. The problem is to bring the system to the steady-state motion $\omega = \omega_0$, $\theta = \theta_0$, where ω_0 is the prescribed angular velocity of rotation around the axis $\{O_{x2}O\}$. The initial angular velocity $\omega(0)$ and the initial magnitudes of $\theta(0)$ and $\theta'(0)$ are assumed to be close to the prescribed values ω_0, θ_0, and $\theta'_0 = 0$.

In the spherical coordinates $\rho = r_0 = \text{const}$, θ, and ϕ we have

$$T = {}^1/_2 m r_0^2\ ([\theta']^2 + \sin^2\theta\ [\varphi']^2) \tag{5.1}$$

$$\Pi = -\ m r_0 g \cos\theta \tag{5.2}$$

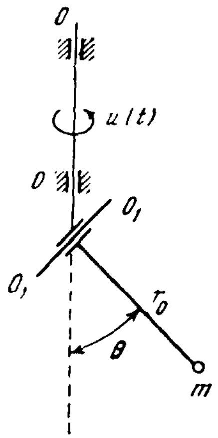

FIG. 4.

The coordinate ϕ is cyclic and equation (1.2) has the form

$$\theta'' - \frac{p^2 \cos\theta}{m^2 r_0^4 \sin^3\theta} + g\frac{\sin\theta}{r_0} = 0,$$

$$p' = u \qquad (p = mr_0^2 \sin^2\theta\,\varphi') \tag{5.3}$$

In the prescribed steady-state motion

$$p = mr_0^2\omega_0 \sin^2\theta_0,$$

$$\cos\theta_0 = g / \omega_0^2 r_0, \qquad (g < \omega_0^2 r_0) \tag{5.4}$$

Let us set up the equations of perturbed motion for system (5.3) in the neighborhood of motion (5.4). We get

$$\Delta\theta'' + \Delta\theta\omega_0^2 (1 + 3\cos^2\theta_0) - \frac{2\omega_0}{mr_0^2}\cot\theta_0 + \nu(\Delta\theta, \Delta p) = 0, \qquad \Delta p' = u \tag{5.5}$$

Here, the expansion of the quantity $\nu(\Delta\theta, \Delta p)$ in powers of $\Delta\theta$ and Δp begins with the second-order terms

$$\nu(\Delta\theta, \Delta p) = -{}^3/_2\,\omega_0^2 (3 + \cos^2\theta_0)\cot\theta_0 \Delta\theta^2 +$$
$$+ \frac{2\omega_0 (1 + \cos^2\theta_0)}{mr_0^2 \sin^2\theta_0}\Delta\theta\Delta p - \frac{\cos\theta_0}{m^2 r_0^4 \sin^3\theta_0}\Delta p^2 + \ldots$$

By setting

$$x_1 = \Delta\theta, \quad x_2 = \Delta\theta' / \omega_0 \sqrt{1 + 3\cos^2\theta_0}, \quad x_3 = \Delta p$$

and by changing the time scale to

$$\omega_0 \sqrt{1 + 3\cos^2\theta_0}\, t = \tau,$$

we reduce the linear part of system (5.5) to the form of (2.1)

$$\frac{dx_1}{d\tau} = x_2, \qquad \frac{dx_2}{d\tau} = -x_1 + \gamma x_3, \qquad \frac{dx_3}{d\tau} = \beta u$$
$$(\gamma = 2\cos\theta_0 / mr_0^2\omega_0 (1 + 3\cos^2\theta_0)\sin\theta_0, \qquad \beta = 1 / \omega_0\sqrt{1 + 3\cos^2\theta_0}) \tag{5.6}$$

The quantities β and γ are nonzero. Consequently, system (5.6) is completely controllable and the problem has a solution. The τ^0 in which it is required to effect the control is taken equal to one period of the natural oscillations of system (5.6), i.e., $\tau^0 = 2\pi$. (In the original time scale of t, the control time is

$$t^{\circ} = 2\pi / \omega_0 \sqrt{1 + 3\cos^2\theta_0}\ .)$$

$$g = 10 \text{ m/sec},\ m = 1 \text{ kg},\ r_0 = 0.4 \text{ m},\ \omega_0 = 10 \text{ sec}^{-1},$$

$$\theta_0 = 1.3.8 \text{ rad},\ \theta'_0 = 0,\ \omega(0) = 8 \text{ sec}^{-1},\ \theta(0) = 1.518 \text{ rad},$$

$$\theta'(0) = 0.$$

Then, in system (5.6) we have $\gamma = 0.272$, $\beta = 0.093$, the fundamental matrix of the homogeneous system has the form (3.4), the initial conditions are $x_{10} = 0.200$, $x_{20} = 0$, and $x_{30} = -0.224$, the vector $c = \{-x_{10}, -x_{20}, -x_{30}\} = \{-0.200, 0, 0.224\}$, and the plane (3.9) is defined by the vector $c^* = \{-c_2 / \beta\gamma,\ (\gamma c_3 - c_1) / \beta\gamma,\ c_3 / \beta\} = \{0,\ 10.455,\ 2.436\}$. After finding the values of λ_i° graphically, the control (3.13) which solves the linear problem (5.6) has the form

$$u_{(1)}^{\circ}(t, x^{\circ}) = 2.672 \text{ sign } (0.244 \cos t + 0.054) \tag{5.7}$$

The control (5.7) is a relay function changing sign when $t_1 = 1.795$ sec, and $t_2 = 4.488$ sec. The control $u_{(1)}^{\circ}(t, x^{\circ})$ in (5.7) transfers system (5.6) to the point $x(2\pi) = 0$ in the time $\tau^{\circ} = 2\pi$ along a trajectory described by the equations

$$\begin{aligned} x_1(t, x_0^{\circ}) &= -0.061 + 0.261 \cos t + \int_0^t 0.025 u_{(1)}^{\circ}(\vartheta, x^{\circ})(1 - \cos(t - \vartheta))\, d\vartheta \\ x_2(t, x^{\circ}) &= -0.261 \sin t + \int_0^t 0.025 u_{(1)}^{\circ}(\vartheta, x^{\circ}) \sin(t - \vartheta)\, d\vartheta \\ x_3(t, x^{\circ}) &= -0.224 + \int_0^t 0.093 u_{(1)}^{\circ}(\vartheta, x^{\circ})\, d\vartheta \end{aligned} \tag{5.8}$$

Taking into account the second-order terms

$$f^{(2)}(x, u) = \{0,\ ax_1^2 + bx_1x_3 + cx_3^2,\ 0\}\ (a = 0.999,\ b = -1.263,\ c = 0.091)$$

system (5.5) is brought to the state $y^{(1)} = \{-0.058,\ 0.026,\ 0\}$, as determined by equation (4.5). Solving the problem of bringing the system (5.6) to the point $-y^{(1)}$ we find

$$c_{(1)} = \{-0.142,\ -0.026,\ 0.224\}, \qquad c_{(1)}^* = \{1.027,\ 8.140,\ 2.436\}$$

The control $u_{(2)}^{\circ}(t, x^{\circ})$ eqpals

$$u_{(2)}^{\circ}(t, x^{\circ}) = 2.136 \text{ sign } (0.030 \sin t + 0.234 \cos t + 0.082) \tag{5.9}$$

With due regard to second-order terms, the control (5.9) transfers the system to the point $\{-0.007, 0.006, 0\}$ along a trajectory computable by (5.8) where instead of $u_{(1)}^{\circ}(\theta, x^{\circ})$ we must substitute $u_{(2)}^{\circ}(\theta, x^{\circ})$ in accordance with (5.9).

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